

Another weighted approximation of functions with singularities by combinations of Bernstein operators

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Abstract

A new type of combinations of Bernstein operators is given in [11]. Here, we introduce another one, which can be used to approximate the functions with singularities. The direct and inverse results of the weighted approximation of this new type combinations are given.

Keywords: Combinations of modified Bernstein polynomials; Functions with singularities; Weighted approximation; Direct and inverse results

1 Introduction

The present work continues to study modified Bernstein operators following [11]. Here, the notations are referred to [11]. For convenience, these notations will be listed. The set of all continuous functions, defined on the interval I , is denoted by $C(I)$. For any $f \in C([0, 1])$, the corresponding Bernstein operators are defined as follows:

$$B_n(f, x) := \sum_{k=0}^n f\left(\frac{k}{n}\right) p_{nk}(x),$$

where

$$p_{nk}(x) := \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, 1, 2, \dots, n, \quad x \in [0, 1].$$

Approximation properties of Bernstein operators have been studied very well (see [2], [3], [5]-[8], [12]-[14], for example). In order to approximate the functions with singularities, Della Vecchia et al. [3] and Yu-Zhao [12] introduced some kinds of modified Bernstein operators.

Let

$$\bar{w}(x) = |x - \xi|^\alpha, \quad 0 < \xi < 1, \quad \alpha > 0,$$

and

$$C_{\bar{w}} := \{f \in C([0, 1] \setminus \xi) : \lim_{x \rightarrow \xi} (\bar{w}f)(x) = 0\}.$$

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The norm in $C_{\bar{w}}$ is defined as $\|f\|_{C_{\bar{w}}} := \|\bar{w}f\| = \sup_{0 \leq x \leq 1} |(\bar{w}f)(x)|$. Define

$$W_{\bar{w}}^r := \{f \in C_{\bar{w}} : f^{(r-1)} \in A.C.((0, 1)), \|\bar{w}\varphi^r f^{(r)}\| < \infty\}.$$

For $f \in C_{\bar{w}}$, define the weighted modulus of smoothness by

$$\omega_{\varphi}^r(f, t)_{\bar{w}} := \sup_{0 < h \leq t} \{\|\bar{w}\Delta_{h\varphi}^r f\|_{[16h^2, 1-16h^2]} + \|\bar{w}\vec{\Delta}_h^r f\|_{[0, 16h^2]} + \|\bar{w}\overleftarrow{\Delta}_h^r f\|_{[1-16h^2, 1]}\},$$

where

$$\Delta_{h\varphi}^r f(x) = \sum_{k=0}^r (-1)^k \binom{r}{k} f(x + (\frac{r}{2} - k)h\varphi(x)),$$

$$\vec{\Delta}_h^r f(x) = \sum_{k=0}^r (-1)^k \binom{r}{k} f(x + (r - k)h),$$

$$\overleftarrow{\Delta}_h^r f(x) = \sum_{k=0}^r (-1)^k \binom{r}{k} f(x - kh),$$

and $\varphi(x) = \sqrt{x(1-x)}$. The weighted K -function is given by

$$K_{r,\varphi}(f, t^r)_{\bar{w}} := \inf_g \{\|\bar{w}(f - g)\| + t^r \|\bar{w}\varphi^r g^{(r)}\| : g \in W_{\bar{w}}^r\}.$$

It was shown in [5] that $K_{\varphi}(f, t^r)_{\bar{w}} \sim \omega_{\varphi}^r(f, t)_{\bar{w}}$. Della Vecchia et al. firstly introduced $B_n^*(f, x)$ and $\bar{B}_n(f, x)$ in [3], where the properties of $B_n^*(f, x)$ and $\bar{B}_n(f, x)$ are studied. Among others, they prove that

$$\|w(f - B_n^*(f))\| \leq C\omega_{\varphi}^2(f, n^{-1/2}), \quad f \in C_w,$$

$$\|\bar{w}(f - \bar{B}_n(f))\| \leq \frac{C}{n^{3/2}} \sum_{k=1}^{[\sqrt{n}]} k^2 \omega_{\varphi}^2(f, \frac{1}{k})_{\bar{w}}^*, \quad f \in C_{\bar{w}},$$

where $w(x) = x^{\alpha}(1-x)^{\beta}$, $\alpha, \beta \geq 0$, $\alpha + \beta > 0$, $0 \leq x \leq 1$. In [11], for any $\alpha, \beta > 0$, $n \geq 2r + \alpha + \beta$, there hold

$$\|wB_{n,r}^*(f)\| \leq C\|wf\|, \quad f \in C_w,$$

$$\|w(B_{n,r}^*(f) - f)\| \leq \begin{cases} \frac{C}{n^r} (\|wf\| + \|w\varphi^{2r} f^{(2r)}\|), & f \in W_w^{2r}, \\ C(\omega_{\varphi}^{2r}(f, n^{-1/2})_w + n^{-r}\|wf\|), & f \in C_w, \end{cases}$$

$$\|w\varphi^{2r} B_{n,r}^{*(2r)}(f)\| \leq \begin{cases} Cn^r \|wf\|, & f \in C_w, \\ C(\|wf\| + \|w\varphi^{2r} f^{(2r)}\|), & f \in W_w^{2r}. \end{cases}$$

and for $0 < \gamma < 2r$,

$$\|w(B_{n,r}^*(f) - f)\| = O(n^{-\gamma/2}) \iff \omega_{\varphi}^{2r}(f, t)_w = O(t^r).$$

On the other hand, since the Bernstein polynomials cannot be used for the investigation of higher orders of smoothness, Butzer [1] introduced the combinations of Bernstein polynomials which have higher orders of approximation. Ditzian and Totik [5] extended this method of combinations and defined the following combinations of Bernstein operators:

$$B_{n,r}(f, x) := \sum_{i=0}^{r-1} C_i(n) B_{n_i}(f, x) \tag{1.1}$$

with the conditions

- (a) $n = n_0 < n_1 < \dots < n_{r-1} \leq Cn$,
- (b) $\sum_{i=0}^{r-1} |C_i(n)| \leq C$,
- (c) $\sum_{i=0}^{r-1} C_i(n) = 1$,
- (d) $\sum_{i=0}^{r-1} C_i(n)n_i^{-k} = 0$, for $k = 1, \dots, r-1$.

Some approximation behaviors of the operators defined as (1.1) can be found in [4]-[6], [9] and [10]. For example, Ditzian and Totik [5] showed that

$$\|B_{n,r}(f) - f\| \leq C(\omega_\varphi^{2r}(f, n^{-1/2}) + n^{-r}\|f\|),$$

and for $0 < \alpha < 2r$,

$$\|B_{n,r}(f) - f\| = O(n^{-\alpha/2}) \iff \omega_\varphi^{2r}(f, n^{-1/2}) = O(t^\alpha),$$

where $\omega_\varphi^{2r}(f, t)$ is the modulus of smoothness with the step-weight function $\varphi(x)$, and $\|f\| = \|f\|_{C([0,1])}$.

The main purpose of the present paper is to give another new type of combinations of Bernstein operators (combinations defined as (1.1) cannot be used to approximate functions in $C_{\bar{w}}$) so as to obtain higher approximation order. In Section 2, we will give the new type of combinations, and the direct and inverse results of the weighted approximation by the new type of combinations. Some lemmas will be given in Section 3, while the proofs of the results will be given in Section 4. Throughout the paper, C denotes a positive constant independent of n and x , which may be different in different cases.

2 The main results

For any positive integer r , we consider the determinant

$$A_r := \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 2r+1 & 2r+2 & 2r+3 & \dots & 4r+1 \\ (2r)(2r+1) & (2r+1)(2r+2) & (2r+2)(2r+3) & \dots & (4r)(4r+1) \\ \dots & \dots & \dots & \ddots & \dots \\ 2 \dots (2r+1) & 3 \dots (2r+2) & 4 \dots (2r+3) & \dots & (2r+2) \dots (4r+1) \end{vmatrix}.$$

We obtain $A_r = \prod_{j=2}^{2r} j!$. Thus, there is a unique solution for the system of nonhomogeneous linear equations:

$$\begin{cases} a_1 + a_2 + \dots + a_{2r+1} = 1, \\ (2r+1)a_1 + (2r+2)a_2 + \dots + (4r+1)a_{2r+1} = 0, \\ (2r+1)(2r)a_1 + (2r+1)(2r+2)a_2 + \dots + (4r)(4r+1)a_{2r+1} = 0, \\ \vdots \\ (2r+1)!a_1 + 3 \dots (2r+2)a_2 + \dots + (2r+2) \dots (4r+1)a_{2r+1} = 0. \end{cases} \quad (2.1)$$

Let

$$\psi(x) = \begin{cases} a_1 x^{2r+1} + a_2 x^{2r+2} + \dots + a_{2r+1} x^{4r+1}, & 0 < x < 1, \\ 0, & x \leq 0, \\ 1, & x = 1. \end{cases}$$

with the coefficients $a_1, a_2, \dots, a_{2r+1}$ satisfying (2.1). From (2.1), we see that $\psi(x) \in C^{(2r)}(-\infty, +\infty)$, $0 \leq \psi(x) \leq 1$ for $0 \leq x \leq 1$. Moreover, it holds that $\psi(1) = 1$, $\psi^{(i)}(0) =$

0, $i = 0, 1, \dots, 2r$. and $\psi^{(i)}(1) = 0$, $i = 1, 2, \dots, 2r$.

Let

$$H(f, x) := \sum_{i=1}^{r+1} f(x_i) l_i(x),$$

and

$$l_i(x) := \frac{\prod_{j=1, j \neq i}^{r+1} (x - x_j)}{\prod_{j=1, j \neq i}^{r+1} (x_i - x_j)}, \quad x_i = \frac{[n\xi - ((r-1)/2 + i)]}{n}, \quad i = 1, 2, \dots, r+1.$$

Further, let

$$x'_1 = \frac{[n\xi - 2\sqrt{n}]}{n}, \quad x'_2 = \frac{[n\xi - \sqrt{n}]}{n}, \quad x'_3 = \frac{[n\xi + \sqrt{n}]}{n}, \quad x'_4 = \frac{[n\xi + 2\sqrt{n}]}{n},$$

and

$$\bar{\psi}_1(x) = \psi\left(\frac{x - x'_1}{x'_2 - x'_1}\right), \quad \bar{\psi}_2(x) = \psi\left(\frac{x - x'_3}{x'_4 - x'_3}\right).$$

Set

$$\bar{F}_n(f, x) := \bar{F}_n(x) = f(x)(1 - \bar{\psi}_1(x) + \bar{\psi}_2(x)) + \bar{\psi}_1(x)(1 - \bar{\psi}_2(x))H(x).$$

We have

$$\bar{F}_n(f, x) = \begin{cases} f(x), & x \in [0, x_{r-5/2}] \cup [x_{r+3/2}, 1], \\ f(x)(1 - \bar{\psi}_1(x)) + \bar{\psi}_1(x)H(x), & x \in [x_{r-5/2}, x_{r-3/2}], \\ H(x), & x \in [x_{r-3/2}, x_{r+1/2}], \\ H(x)(1 - \bar{\psi}_2(x)) + \bar{\psi}_2(x)f(x), & x \in [x_{r+1/2}, x_{r+3/2}]. \end{cases}$$

Obviously, $\bar{F}_n(f, x)$ is linear, reproduces polynomials of degree r , and $\bar{F}_n(f, x) \in C^{(2r)}([0, 1])$, provided that $f \in C^{(2r)}([0, 1])$.

Now, we can define our new combinations of Bernstein operators as follows:

$$\bar{B}_{n,r}(f, x) := B_{n,r}(\bar{F}_n, x) = \sum_{i=0}^{r-1} C_i(n) B_{n_i}(\bar{F}_n, x),$$

where $C_i(n)$ satisfy the conditions (a)-(d). Our main result is the following:

Theorem. For any $\alpha > 0$, $0 \leq \lambda \leq 1$, we have

$$\|\bar{w} \bar{B}_{n,r}^{(2r)}(f)\| \leq C n^{2r} \|\bar{w} f\|, \quad f \in W_{\bar{w}}^{2r}, \quad (2.2)$$

$$|\bar{w}(x) \varphi^{2r\lambda}(x) \bar{B}_{n,r}^{(2r)}(f, x)| \leq \begin{cases} C n^r \{\max\{n^{r(1-\lambda)}, \varphi^{2r(\lambda-1)}\}\} \|\bar{w} f\|, & f \in C_{\bar{w}}, \\ C(\|\bar{w} f\| + \|\bar{w} \varphi^{2r\lambda} f^{(2r)}\|), & f \in W_{\bar{w}}^{2r}, \end{cases} \quad (2.3)$$

$$\|\bar{w} \bar{B}_{n,r}(f)\| \leq C \|\bar{w} f\|, \quad f \in C_{\bar{w}}, \quad (2.4)$$

$$\|\bar{w}(\bar{B}_{n,r}(f) - f)\| \leq \begin{cases} \frac{C}{n^r} (\|\bar{w} f\| + \|\bar{w} \varphi^{2r} f^{(2r)}\|), & f \in W_{\bar{w}}^{2r}, \\ C(\omega_{\varphi}^{2r}(f, n^{-1/2})_{\bar{w}} + n^{-r} \|\bar{w} f\|), & f \in C_{\bar{w}}, \end{cases} \quad (2.5)$$

and for $0 < \gamma < 2r$,

$$\|\bar{w}(\bar{B}_{n,r}(f) - f)\| = O(n^{-\gamma/2}) \iff \omega_{\varphi}^{2r}(f, t)_{\bar{w}} = O(t^r). \quad (2.6)$$

3 Lemmas

Lemma 1. ([13]) For any non-negative real u and v , we have

$$\sum_{k=1}^{n-1} \left(\frac{k}{n}\right)^{-u} \left(1 - \frac{k}{n}\right)^{-v} p_{nk}(x) \leq C x^{-u} (1-x)^{-v}. \quad (3.1)$$

Lemma 2. For any positive real α , and $f \in W_{\bar{w}}^{2r}$, we have

$$\|\bar{w}\varphi^{2r-2j} f^{(2r-j)}\| \leq C(\|\bar{w}f\| + \|\bar{w}\varphi^{2r} f^{(2r)}\|). \quad (3.2)$$

Proof. *Case 1.* $\xi \in [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$. It follows from Kolmogolov's inequality that

$$|f^{(2r-j)}(\frac{1}{2})| \leq C(\|f\|_{[1/4, 3/4]} + \|f^{(2r)}\|_{[1/4, 3/4]}),$$

Moreover,

$$|f^{(2r-j)}(\frac{1}{2})| \leq C(\|\bar{w}f\|_{[1/4, 3/4]} + \|\bar{w}\varphi^{2r} f^{(2r)}\|_{[1/4, 3/4]}). \quad (3.3)$$

When $0 \leq x \leq \frac{1}{2}$, u between x and $\frac{k}{n}$, we have $\frac{|k/n-u|^{r-1}}{\bar{w}(u)} \leq \frac{|k/n-x|^{r-1}}{\bar{w}(x)}$, then

$$\begin{aligned} |f^{(2r-j)}(x) - f^{(2r-j)}(\frac{1}{2})| &\leq \int_x^{\frac{1}{2}} |f^{(2r-j+1)}(u)| du \\ &\leq C \|\bar{w}\varphi^{2r-2j+2} f^{(2r-j+1)}\| \int_x^{\frac{1}{2}} \frac{du}{\bar{w}(u)\varphi^{2r-2j+2}(u)} \\ &= C \|\bar{w}\varphi^{2r-2j+2} f^{(2r-j+1)}\| \int_x^{\frac{1}{2}} \frac{|k/n-u|^{r-1} du}{|k/n-u|^{r-1} \bar{w}(u)\varphi^{2r-2j+2}(u)} \\ &\leq C \|\bar{w}\varphi^{2r-2j+2} f^{(2r-j+1)}\| \left(\int_x^{\frac{1}{2}} \frac{|k/n-u|^{r-1} du}{|k/n-x|^{r-1} \bar{w}(u)\varphi^{2r-2j+2}(u)} \right. \\ &\quad \left. + \int_x^{\frac{1}{2}} \frac{|k/n-u|^{r-1} du}{|k/n-1/2|^{r-1} \bar{w}(u)\varphi^{2r-2j+2}(u)} \right) \\ &\leq C \|\bar{w}\varphi^{2r-2j+2} f^{(2r-j+1)}\| \frac{x^{-r+j}}{\bar{w}(x)}, \end{aligned}$$

which, together with (3.3), gives that

$$|\bar{w}(x)\varphi^{2r-2j}(x)f^{(2r-j)}(x)| \leq C(\|\bar{w}\varphi^{2r-2j+2} f^{(2r-j+1)}\| + \|\bar{w}f\| + \|\bar{w}\varphi^{2r} f^{(2r)}\|).$$

Similarly, we can prove that the above inequality also holds when $1/2 < x \leq 1$. Therefore, we obtain that

$$|\bar{w}(x)\varphi^{2r-2j}(x)f^{(2r-j)}(x)| \leq C(\|\bar{w}\varphi^{2r-2j+2} f^{(2r-j+1)}\| + \|\bar{w}f\| + \|\bar{w}\varphi^{2r} f^{(2r)}\|). \quad (3.4)$$

Now, the result follows from (3.4) when $j = 1$, and thus the result can be deduced from (3.4) by induction when $1 < j \leq r$.

Case 2. $\xi \in [\frac{1}{4}, \frac{3}{4}] \cup \{\frac{1}{2}\}$. The situation goes similarly.

Lemma 3. For any $f \in W_{\bar{w}}^{2r}$, we have

$$\|\bar{w}(f - H)\|_{[x_{r-5/2}, x_{r+3/2}]} \leq \frac{C}{n^r} (\|\bar{w}f\| + \|\bar{w}\varphi^{2r} f^{(2r)}\|). \quad (3.5)$$

Proof. By Taylor expansion, we have

$$f(x_i) = \sum_{u=0}^r \frac{(x_i - x)^u}{u!} f^{(u)}(x) + \frac{1}{r!} \int_x^{x_i} (x_i - s)^r f^{(r+1)}(s) ds, \quad (3.6)$$

It follows from (3.6) and the identity

$$\sum_{i=1}^{r+1} x_i^v l_i(x) = x^v, \quad v = 0, 1, \dots, r.$$

we have

$$\begin{aligned} H(f, x) &= \sum_{i=0}^r \sum_{u=0}^r \frac{(x_i - x)^u}{u!} f^{(u)}(x) l_i(x) + \frac{1}{r!} \sum_{i=1}^{r+1} l_i(x) \int_x^{x_i} (x_i - s)^r f^{(r+1)}(s) ds \\ &= f(x) + \sum_{u=1}^r f^{(u)}(x) \left(\sum_{v=0}^u C_u^v (-x)^{u-v} \sum_{i=1}^r x_i^v l_i(x) \right) \\ &\quad + \frac{1}{r!} \sum_{i=1}^{r+1} l_i(x) \int_x^{x_i} (x_i - s)^r f^{(r+1)}(s) ds, \end{aligned}$$

which implies that

$$\bar{w}(x) |f(x) - H(f, x)| = \frac{1}{r!} \bar{w}(x) \sum_{i=1}^{r+1} l_i(x) \int_x^{x_i} (x_i - s)^r f^{(r+1)}(s) ds,$$

since $|l_i(x)| \leq C$ for $x \in [0, \frac{1}{n}]$, $i = 1, 2, \dots, r+1$.

It follows from $\frac{|x_i - s|^{r-1}}{\bar{w}(s)} \leq \frac{|x_i - x|^{r-1}}{\bar{w}(x)}$, s between x_i and x , then

$$\begin{aligned} \bar{w}(x) |f(x) - H(f, x)| &= C \frac{\bar{w}(x)}{n^r} \sum_{i=1}^{r+1} \int_x^{x_i} |f^{(r+1)}(s)| ds \\ &\leq C \frac{\bar{w}(x)}{n^r} \|\bar{w} \varphi^2 f^{(r+1)}\| \sum_{i=1}^{r+1} \int_x^{x_i} |\bar{w}^{-1}(s) \varphi^{-2}(s)| ds \\ &\leq \frac{C}{n^r} \|\bar{w} \varphi^2 f^{(r+1)}\|. \end{aligned}$$

which, together with (3.2) (when $j = r - 1$) implies (3.5).

Lemma 4. For any $f \in W_{\bar{w}}^{2r}$ and $\alpha > 0$, we have

$$\|\bar{w} \varphi^{2r} \bar{F}_n^{(2r)}\| \leq C (\|\bar{w} \varphi^{2r} f^{(2r)}\| + \|\bar{w} f\|). \quad (3.7)$$

Proof. We only prove the above result when $x \in [x_{r-5/2}, x_{r-3/2}]$, the others can be done similarly. Obviously,

$$\begin{aligned} |\bar{F}_n^{(2r)}(x)| &= |(H(x) + \bar{\psi}_1(x)(f(x) - H(x)))^{(2r)}| \\ &= \left| \sum_{i=0}^{2r} C_{2r}^i (\bar{\psi}_1(x))^{(i)} (f(x) - H(x))^{(2r-i)} \right| \\ &\leq C \sum_{i=0}^{2r} n^{\frac{i}{2}} |(f(x) - H(x))^{(2r-i)}| \end{aligned}$$

If $2r - i \geq r + 1$, using (3.2), then

$$\begin{aligned} |\bar{w}(x)\varphi^{2r}(x)(f(x) - H(x))^{(2r-i)}| &= |\bar{w}(x)\varphi^{2r-2i}(x)f^{(2r-i)}(x)| \cdot \varphi^{2i}(x) \\ &\leq Cn^{-i}(\|\bar{w}\varphi^{2r}f^{(2r)}\| + \|\bar{w}f\|). \end{aligned}$$

If $0 < 2r - i < r + 1$, by the following well-known inequality

$$\|g^{(j)}\| \leq C((d-c)^{-j}\|g\|_{[c,d]} + (d-c)^{(2r-j)}\|g^{(2r)}\|_{[c,d]}), \quad 0 < j < 2r,$$

we get

$$\begin{aligned} &|\bar{w}(x)\varphi^{2r}(x)(f(x) - H(x))^{(2r-i)}| \\ &\leq C\bar{w}(x)\varphi^{2r}(x)(n^{2r-i}\|f - H\|_{[x_{r-5/2}, x_{r-3/2}]} + n^{-i}\|f^{(2r)}\|_{[x_{r-5/2}, x_{r-3/2}]}) \\ &\leq C(n^{r-i}\|\bar{w}(f - H)\|_{[x_{r-5/2}, x_{r-3/2}]} + n^{-i}\|\bar{w}\varphi^{2r}f^{(2r)}\|_{[x_{r-5/2}, x_{r-3/2}]}) \\ &\leq Cn^{-i}(\|\bar{w}\varphi^{2r}f^{(2r)}\| + \|\bar{w}f\|). \end{aligned}$$

If $i = 2r$, by (3.5), we have

$$\begin{aligned} |\bar{w}(x)\varphi^{2r}(x)(f(x) - H(x))^{(2r-i)}| &= |\bar{w}(x)\varphi^{2r}(x)(f(x) - H(x))| \\ &\leq Cn^{-2r}(\|\bar{w}\varphi^{2r}f^{(2r)}\| + \|\bar{w}f\|). \end{aligned}$$

Now the lemma follows from bringing these results together.

Lemma 5. Let $A_n(x) := \bar{w}(x) \sum_{|k-n\xi| \leq \sqrt{n}} p_{n,k}(x)$. Then $A_n(x) \leq Cn^{-\alpha/2}$ for $0 < \xi < 1$ and $\alpha > 0$.

Proof. If $|x - \xi| \leq \frac{3}{\sqrt{n}}$, then the statement is trivial. Hence assume $0 \leq x \leq \xi - \frac{3}{\sqrt{n}}$ (the case $\xi + \frac{3}{\sqrt{n}} \leq x \leq 1$ can be treated similarly). Then for a fixed x the maximum of $p_{n,k}(x)$ is attained for $k = k_n := [n\xi - \sqrt{n}]$. By using Stirling's formula, we get

$$\begin{aligned} p_{n,k_n}(x) &\leq C \frac{\left(\frac{n}{e}\right)^n \sqrt{n} x^{k_n} (1-x)^{n-k_n}}{\left(\frac{k_n}{e}\right)^{k_n} \sqrt{k_n} \left(\frac{n-k_n}{e}\right)^{n-k_n} \sqrt{n-k_n}} \\ &\leq \frac{C}{\sqrt{n}} \left(\frac{nx}{k_n}\right)^{k_n} \left(\frac{n(1-x)}{n-k_n}\right)^{n-k_n} \\ &= \frac{C}{\sqrt{n}} \left(1 - \frac{k_n - nx}{k_n}\right)^{k_n} \left(1 + \frac{k_n - nx}{n-k_n}\right)^{n-k_n}. \end{aligned}$$

Now from the inequalities

$$k_n - nx = [n\xi - \sqrt{n}] - nx > n(\xi - x) - \sqrt{n} - 1 \geq \frac{1}{2}n(\xi - x),$$

and

$$1 - u \leq e^{-u - \frac{1}{2}u^2}, \quad 1 + u \leq e^u, \quad u \geq 0.$$

We have that the second inequality is valid. To prove the first one we consider the function $\lambda(u) = e^{-u - \frac{1}{2}u^2} + u - 1$. Here $\lambda(0) = 0$, $\lambda'(u) = -(1+u)e^{-u - \frac{1}{2}u^2} + 1$, $\lambda'(0) = 0$, $\lambda''(u) = u(u+2)e^{-u - \frac{1}{2}u^2} \geq 0$, whence $\lambda(u) \geq 0$ for $u \geq 0$. Hence

$$\begin{aligned} p_{n,k_n}(x) &\leq \frac{C}{\sqrt{n}} \exp\left\{k_n \left[-\frac{k_n - nx}{k_n} - \frac{1}{2} \left(\frac{k_n - nx}{k_n}\right)^2\right] + k_n - nx\right\} \\ &= \frac{C}{\sqrt{n}} \exp\left\{\frac{(k_n - nx)^2}{2k_n}\right\} \leq e^{-Cn(\xi-x)^2}. \end{aligned}$$

Thus $A_n(x) \leq C(\xi - x)^\alpha e^{-Cn(\xi - x)^2}$. An easy calculation shows that here the maximum is attained when $\xi - x = \frac{C}{\sqrt{n}}$ and the lemma follows.

Lemma 6. For $0 < \xi < 1$, $\alpha, \beta > 0$, we have

$$\bar{w}(x) \sum_{|k - n\xi| \leq \sqrt{n}} |k - nx|^\beta p_{n,k}(x) \leq Cn^{(\beta - \alpha/2)} \varphi^\beta(x). \quad (3.8)$$

Proof. By (3.1) and the lemma 5, we have

$$\bar{w}(x)^{\frac{1}{2n}} (\bar{w}(x) \sum_{|k - n\xi| \leq \sqrt{n}} p_{n,k}(x))^{\frac{2n-1}{2n}} \left(\sum_{|k - n\xi| \leq \sqrt{n}} |k - nx|^{2n\beta} p_{n,k}(x) \right)^{\frac{1}{2n}} \leq Cn^{(\beta - \alpha/2)} \varphi^\beta(x).$$

4 Proof of Theorem 1

4.1 Proof of (2.2)

We first prove $x \in [0, \frac{1}{n}]$ (The same as $x \in [1 - \frac{1}{n}, 1]$), now

$$\begin{aligned} |\bar{w}(x) \bar{B}_{n,r}^{(2r)}(f, x)| &\leq \bar{w}(x) \sum_{i=0}^{r-1} \frac{n_i!}{(n_i - 2r)!} \sum_{k=0}^{n_i - 2r} |C_i(n) \vec{\Delta}_{\frac{1}{n_i}}^{2r} \bar{F}_n(\frac{k}{n_i})| p_{n_i - 2r, k}(x) \\ &\leq C \bar{w}(x) \sum_{i=0}^{r-1} n_i^{2r} \sum_{k=0}^{n_i - 2r} |C_i(n) \vec{\Delta}_{\frac{1}{n_i}}^{2r} \bar{F}_n(\frac{k}{n_i})| p_{n_i - 2r, k}(x) \\ &\leq C \bar{w}(x) \sum_{i=0}^{r-1} n_i^{2r} \sum_{k=0}^{n_i - 2r} \sum_{j=0}^{2r} C_{2r}^j |C_i(n) \bar{F}_n(\frac{k + 2r - j}{n_i})| p_{n_i - 2r, k}(x) \\ &\leq C \bar{w}(x) \sum_{i=0}^{r-1} n_i^{2r} \sum_{j=0}^{2r} C_{2r}^j |C_i(n) \bar{F}_n(\frac{2r - j}{n_i})| p_{n_i - 2r, 0}(x) \\ &\quad + C \bar{w}(x) \sum_{i=0}^{r-1} n_i^{2r} \sum_{j=0}^{2r} C_{2r}^j |C_i(n) \bar{F}_n(\frac{n_i - j}{n_i})| p_{n_i - 2r, n_i - 2r}(x) \\ &\quad + C \bar{w}(x) \sum_{i=0}^{r-1} n_i^{2r} \sum_{k=1}^{n_i - 2r - 1} \sum_{j=0}^{2r} C_{2r}^j |C_i(n) \bar{F}_n(\frac{k + 2r - j}{n_i})| p_{n_i - 2r, k}(x) \\ &:= H_1 + H_2 + H_3. \end{aligned} \quad (4.1)$$

We have

$$\begin{aligned} H_1 &\leq C \bar{w}(x) \sum_{i=0}^{r-1} n_i^{2r} \left(\sum_{j=0}^{2r-1} |C_i(n) \bar{F}_n(\frac{2r - j}{n_i})| + |\bar{F}_n(0)| \right) p_{n_i - 2r, 0}(x) \\ &\leq C n^{2r} \|\bar{w}f\| \sum_{i=0}^{r-1} \sum_{j=0}^{2r-1} \left(\frac{n_i |x - \xi|}{2r - j - n_i \xi} \right)^\alpha (1 - x)^{n_i - 2r} \\ &\leq C n^{2r} \|\bar{w}f\| \sum_{i=0}^{r-1} (n_i |x - \xi|)^\alpha (1 - x)^{n_i - 2r} \\ &\leq C n^{2r} \|\bar{w}f\|. \end{aligned}$$

Similarly, we can get $H_2 \leq C n^{2r} \|\bar{w}f\|$ and $H_3 \leq C n^{2r} \|\bar{w}f\|$.

When $x \in [\frac{1}{n}, 1 - \frac{1}{n}]$, according to [5], we have

$$\begin{aligned}
& |\bar{w}(x)\bar{B}_{n,r}^{(2r)}(f, x)| \\
&= |\bar{w}(x)B_{n,r}^{(2r)}(\bar{F}_n, x)| \\
&= \bar{w}(x)(\varphi^2(x))^{-2r} \sum_{i=0}^{r-1} \sum_{j=0}^{2r} |Q_j(x, n_i)C_i(n)|n_i^j \sum_{k/n_i \in A} |(x - \frac{k}{n_i})^j \bar{F}_n(\frac{k}{n_i})|p_{n_i,k}(x) \\
&\quad + \bar{w}(x)(\varphi^2(x))^{-2r} \sum_{i=0}^{r-1} \sum_{j=0}^{2r} |Q_j(x, n_i)C_i(n)|n_i^j \sum_{x'_2 \leq k/n_i \leq x'_3} |(x - \frac{k}{n_i})^j H(\frac{k}{n_i})|p_{n_i,k}(x) \\
&:= \sigma_1 + \sigma_2.
\end{aligned} \tag{4.2}$$

Where $A := [0, x'_2] \cup [x'_3, 1]$, H is a linear function. If $\frac{k}{n_i} \in A$, when $\frac{\bar{w}(x)}{\bar{w}(\frac{k}{n_i})} \leq C(1 + n_i^{-\frac{\alpha}{2}}|k - n_i x|^\alpha)$, we have $|k - n_i x| \geq \frac{\sqrt{n_i}}{2}$, then $Q_j(x, n_i) = (n_i x(1 - x))^{[(2r-j)/2]}$, and $(\varphi^2(x))^{-2r} Q_j(x, n_i) n_i^j \leq C(n_i/\varphi^2(x))^{r+j/2}$. By (3.8), then

$$\begin{aligned}
\sigma_1 &\leq C\bar{w}(x) \sum_{i=0}^{r-1} \sum_{j=0}^{2r} |C_i(n)|(\frac{n_i}{\varphi^2(x)})^{r+j/2} \sum_{k=0}^{n_i} |(x - \frac{k}{n_i})^j \bar{F}_n(\frac{k}{n_i})|p_{n_i,k}(x) \\
&\leq C\|\bar{w}f\| \sum_{i=0}^{r-1} \sum_{j=0}^{2r} |C_i(n)|(\frac{n_i}{\varphi^2(x)})^{r+j/2} \sum_{k=0}^{n_i} [1 + n_i^{-\frac{\alpha}{2}}|k - n_i x|^\alpha] |x - \frac{k}{n_i}|^j p_{n_i,k}(x) \\
&:= I_1 + I_2.
\end{aligned}$$

By a simple calculation, we have $I_1 \leq Cn^{2r}\|\bar{w}f\|$. By (3.1), then

$$I_2 \leq C\|\bar{w}f\| \sum_{i=0}^{r-1} \sum_{j=0}^{2r} |C_i(n)|n_i^{-(\frac{\alpha}{2}+j)} (\frac{n_i}{\varphi^2(x)})^{j/2} \sum_{k=0}^{n_i} |k - n_i x|^{\alpha+j} p_{n_i,k}(x) \leq Cn^{2r}\|\bar{w}f\|.$$

We note that $|H(\frac{k}{n_i})| \leq \max(|H(x'_1)|, |H(x'_4)|) := H(a)$.

If $x \in [x'_1, x'_4]$, we have $\bar{w}(x) \leq \bar{w}(a)$. So, if $x \in [x'_1, x'_4]$, then

$$\sigma_2 \leq C\bar{w}(a)H(a)n^r\varphi^{-2r}(x) \leq Cn^{2r}\|\bar{w}f\|.$$

If $x \notin [x'_1, x'_4]$, then $\bar{w}(a) > n_i^{-\frac{\alpha}{2}}$, we have

$$\sigma_2 \leq C\bar{w}(a)H(a)\varphi^{-2r}(x)\bar{w}(x) \sum_{i=0}^{r-1} C_i(n)n_i^{r+\frac{\alpha}{2}} \sum_{x'_2 \leq k/n_i \leq x'_3} p_{n_i,k}(x) \leq Cn^{2r}\|\bar{w}f\|.$$

It follows from combining the above inequalities (4.1) and (4.2) that the theorem is proved.

4.2 Proof of (2.3)

(1) When $f \in C_{\bar{w}}$, we discuss it as follows:

Case 1. If $0 \leq \varphi(x) \leq \frac{1}{\sqrt{n}}$, by (2.2), we have

$$|\bar{w}(x)\varphi^{2r\lambda}(x)\bar{B}_{n,r}^{(2r)}(f, x)| \leq Cn^{-r\lambda}|\bar{w}(x)\bar{B}_{n,r}^{(2r)}(f, x)| \leq Cn^{r(2-\lambda)}\|\bar{w}f\|. \tag{4.3}$$

Case 2. If $\varphi(x) > \frac{1}{\sqrt{n}}$, we have

$$\begin{aligned}
& |\bar{B}_{n,r}^{(2r)}(f, x)| = |B_{n,r}^{(2r)}(\bar{F}_n, x)| \\
& \leq (\varphi^2(x))^{-2r} \sum_{i=0}^{r-1} \sum_{j=0}^{2r} |Q_j(x, n_i) C_i(n)| n_i^j \sum_{k=0}^{n_i} |(x - \frac{k}{n_i})^j \bar{F}_n(\frac{k}{n_i})| p_{n_i, k}(x), \\
& Q_j(x, n_i) = (n_i x(1-x))^{[(2r-j)/2]}, \text{ and } (\varphi^2(x))^{-2r} Q_j(x, n_i) n_i^j \leq C(n_i/\varphi^2(x))^{r+j/2}. \text{ So} \\
& |\bar{w}(x) \varphi^{2r\lambda}(x) \bar{B}_{n,r}^{(2r)}(f, x)| \\
& \leq C \bar{w}(x) \varphi^{2r\lambda}(x) \sum_{i=0}^{r-1} \sum_{j=0}^{2r} |C_i(n)| (\frac{n_i}{\varphi^2(x)})^{r+j/2} \sum_{k=0}^{n_i} |(x - \frac{k}{n_i})^j \bar{F}_n(\frac{k}{n_i})| p_{n_i, k}(x) \\
& = C \bar{w}(x) \varphi^{2r\lambda}(x) \sum_{i=0}^{r-1} \sum_{j=0}^{2r} |C_i(n)| (\frac{n_i}{\varphi^2(x)})^{r+j/2} \sum_{k/n_i \in A} |(x - \frac{k}{n_i})^j \bar{F}_n(\frac{k}{n_i})| p_{n_i, k}(x) \\
& \quad + C \bar{w}(x) \varphi^{2r\lambda}(x) \sum_{i=0}^{r-1} \sum_{j=0}^{2r} |C_i(n)| (\frac{n_i}{\varphi^2(x)})^{r+j/2} \sum_{x'_2 \leq k/n_i \leq x'_3} |(x - \frac{k}{n_i})^j \bar{F}_n(\frac{k}{n_i})| p_{n_i, k}(x) \\
& := \sigma_1 + \sigma_2. \tag{4.4}
\end{aligned}$$

Where $A := [0, x'_2] \cup [x'_3, 1]$. We can easily get $\sigma_1 \leq n^r \varphi^{2r(\lambda-1)}(x) \|\bar{w}f\|$, $\sigma_2 \leq n^r \varphi^{2r(\lambda-1)}(x) \|\bar{w}f\|$. By bringing these facts (4.3) and (4.4) together, the theorem is proved.

(2) When $f \in W_w^{2r}$, we have

$$B_{n,r}^{(2r)}(\bar{F}_n, x) = \sum_{i=0}^{r-1} C_i(n) n_i^{2r} \sum_{k=0}^{n_i-2r} \vec{\Delta}_{\frac{1}{n_i}}^{2r} \bar{F}_n(\frac{k}{n_i}) p_{n_i-2r, k}(x). \tag{4.5}$$

If $0 < k < n_i - 2r$, we have

$$|\vec{\Delta}_{\frac{1}{n_i}}^{2r} \bar{F}_n(\frac{k}{n_i})| \leq C n_i^{-2r+1} \int_0^{\frac{2r}{n_i}} |\bar{F}_n^{(2r)}(\frac{k}{n_i} + u)| du, \tag{4.6}$$

If $k = 0$, we have

$$|\vec{\Delta}_{\frac{1}{n_i}}^{2r} \bar{F}_n(0)| \leq C n_i^{-r+1} \int_0^{\frac{2r}{n_i}} u^{2r-1} |\bar{F}_n^{(2r)}(u)| du, \tag{4.7}$$

Similarly

$$|\vec{\Delta}_{\frac{1}{n_i}}^{2r} \bar{F}_n(\frac{n_i-2r}{n_i})| \leq C n_i^{-2r+1} \int_{1-\frac{2r}{n_i}}^1 (1-u)^{2r-1} |\bar{F}_n^{(2r)}(u)| du.$$

By (4.5) and (4.6), we have

$$\begin{aligned}
|\bar{w}(x) \varphi^{2r\lambda}(x) \bar{B}_{n,r}^{(2r)}(f, x)| & \leq C \bar{w}(x) \varphi^{2r\lambda}(x) \sum_{i=0}^{r-1} |C_i(n)| n_i^{2r} \sum_{k=0}^{n_i-2r} |\vec{\Delta}_{\frac{1}{n_i}}^{2r} \bar{F}_n(\frac{k}{n_i})| p_{n_i-2r, k}(x) \\
& = C \bar{w}(x) \varphi^{2r\lambda}(x) \sum_{i=0}^{r-1} |C_i(n)| n_i^{2r} \sum_{k=1}^{n_i-2r-1} |\vec{\Delta}_{\frac{1}{n_i}}^{2r} \bar{F}_n(\frac{k}{n_i})| p_{n_i-2r, k}(x) \\
& \quad + C \bar{w}(x) \varphi^{2r\lambda}(x) \sum_{i=0}^{r-1} |C_i(n)| n_i^{2r} |\vec{\Delta}_{\frac{1}{n_i}}^{2r} \bar{F}_n(0)| p_{n_i-2r, 0}(x) \\
& \quad + C \bar{w}(x) \varphi^{2r\lambda}(x) \sum_{i=0}^{r-1} |C_i(n)| n_i^{2r} |\vec{\Delta}_{\frac{1}{n_i}}^{2r} \bar{F}_n(\frac{n_i-2r}{n_i})| p_{n_i-2r, n_i-2r}(x) \\
& := I_1 + I_2 + I_3.
\end{aligned}$$

By (4.6), we have

$$\begin{aligned}
I_1 &\leq C\bar{w}(x)\varphi^{2r\lambda}(x) \sum_{i=0}^{r-1} |C_i(n)|n_i \sum_{k=1}^{n_i-2r-1} \int_0^{\frac{2r}{n_i}} |\bar{F}_n^{(2r)}(\frac{k}{n_i} + u)| dup_{n_i-2r,k}(x) \\
&= C\bar{w}(x)\varphi^{2r\lambda}(x) \sum_{i=0}^{r-1} |C_i(n)|n_i \sum_{k/n_i \in A} \int_0^{\frac{2r}{n_i}} |\bar{F}_n^{(2r)}(\frac{k}{n_i} + u)| dup_{n_i-2r,k}(x) \\
&\quad + C\bar{w}(x)\varphi^{2r\lambda}(x) \sum_{i=0}^{r-1} |C_i(n)|n_i \sum_{x'_2 \leq k/n_i \leq x'_3} \int_0^{\frac{2r}{n_i}} |H_n^{(2r)}(\frac{k}{n_i} + u)| dup_{n_i-2r,k}(x) \\
&:= T_1 + T_2.
\end{aligned}$$

Where $A := [0, x'_2] \cup [x'_3, 1]$, H is a linear function. If $\frac{k}{n_i} \in A$, when $\frac{\bar{w}(x)}{\bar{w}(\frac{k}{n_i})} \leq C(1 + n_i^{-\frac{\alpha}{2}}|k - n_i x|^\alpha)$, we have $|k - n_i x| \geq \frac{\sqrt{n_i}}{2}$, by (3.1) and (3.7), then

$$\begin{aligned}
T_1 &\leq C\|\bar{w}\varphi^{2r\lambda}F^{(2r)}\|\bar{w}(x)\varphi^{2r\lambda}(x) \sum_{i=0}^{r-1} |C_i(n)|n_i \sum_{k/n_i \in A} \int_0^{\frac{2r}{n_i}} \bar{w}^{-1}(\frac{k}{n_i} + u)\varphi^{-2r\lambda}(\frac{k}{n_i} + u) dup_{n_i-2r,k}(x) \\
&\leq C\|\bar{w}\varphi^{2r\lambda}F^{(2r)}\|\varphi^{2r\lambda}(x) \sum_{i=0}^{r-1} |C_i(n)|n_i \sum_{k=0}^{n_i} \int_0^{\frac{2r}{n_i}} [1 + n_i^{-\frac{\alpha}{2}}|k - n_i x|^\alpha]\varphi^{-2r\lambda}(\frac{k}{n_i}) dup_{n_i-2r,k}(x) \\
&\leq C\|\bar{w}\varphi^{2r\lambda}\bar{F}_n^{(2r)}\| \leq C(\|\bar{w}f\| + \|\bar{w}\varphi^{2r\lambda}f^{(2r)}\|).
\end{aligned}$$

Similarly, we can get $T_2 \leq C(\|\bar{w}f\| + \|\bar{w}\varphi^{2r\lambda}f^{(2r)}\|)$. So $I_1 \leq C(\|\bar{w}f\| + \|\bar{w}\varphi^{2r\lambda}f^{(2r)}\|)$ and by (4.7), we have

$$\begin{aligned}
I_2 &\leq C\bar{w}(x)\varphi^{2r\lambda}(x) \sum_{i=0}^{r-1} |C_i(n)|n_i^{2r} |\vec{\Delta}_{\frac{1}{n_i}}^{2r} \bar{F}_n(0)| p_{n_i-2r,0}(x) \\
&\leq C\bar{w}(x)\varphi^{2r\lambda}(x) \sum_{i=0}^{r-1} |C_i(n)|n_i^{r+1} \int_0^{\frac{2r}{n_i}} u^{2r-1} |\bar{F}_n^{(2r)}(u)| dup_{n_i-2r,0}(x) \\
&\leq C\|\bar{w}\varphi^{2r\lambda}\bar{F}_n^{(2r)}\| \sum_{i=0}^{r-1} (n_i x)^{r(1+\lambda)} (1-x)^{r\lambda} \leq C\|\bar{w}\varphi^{2r\lambda}\bar{F}_n^{(2r)}\| \\
&\leq C(\|\bar{w}f\| + \|\bar{w}\varphi^{2r\lambda}f^{(2r)}\|).
\end{aligned}$$

Analogously, $I_3 \leq C(\|\bar{w}f\| + \|\bar{w}\varphi^{2r\lambda}f^{(2r)}\|)$, then the theorem is proved.

Corollary 1. If $\alpha > 0$ and $\lambda = 0$, we have

$$|\bar{w}(x)\bar{B}_{n,r}^{(2r)}(f, x)| \leq \begin{cases} Cn^{2r}\|\bar{w}f\|, & f \in C_{\bar{w}}, \\ C(\|\bar{w}f\| + \|\bar{w}f^{(2r)}\|), & f \in W_{\bar{w}}^{2r}. \end{cases}$$

Corollary 2. If $\alpha > 0$ and $\lambda = 1$, we have

$$|\bar{w}(x)\varphi^{2r}(x)\bar{B}_{n,r}^{(2r)}(f, x)| \leq \begin{cases} Cn^r\|\bar{w}f\|, & f \in C_{\bar{w}}, \\ C(\|\bar{w}f\| + \|\bar{w}\varphi^{2r}f^{(2r)}\|), & f \in W_{\bar{w}}^{2r}. \end{cases}$$

4.3 Proof of (2.4)

$$\begin{aligned}
|\bar{w}(x)\bar{B}_{n,r}(f, x)| &= |\bar{w}(x)B_{n,r}(\bar{F}_n, x)| \leq \bar{w}(x) \sum_{i=0}^{r-1} \sum_{k=1}^{n_i-1} |C_i(n)\bar{F}_n(\frac{k}{n_i})| p_{n_i,k}(x) \\
&\quad + \bar{w}(x) \sum_{i=0}^{r-1} |C_i(n)\bar{F}_n(0)| p_{n_i,0}(x) + \bar{w}(x) \sum_{i=0}^{r-1} |C_i(n)\bar{F}_n(1)| p_{n_i,n_i}(x) \\
&:= I_1 + I_2 + I_3.
\end{aligned}$$

Analogously, the theorem can be proved easily.

4.4. Proof of (2.5)

We assume $f \in W_{\bar{w}}^{2r}$, then $\|\bar{w}(\bar{B}_{n,r}(f) - \bar{F}_n)\| \leq \frac{C}{n^r}(\|\bar{w}f\| + \|\bar{w}\varphi^{2r}f^{(2r)}\|)$. Recall that [5], then

$$B_{n,r}((t-x)^j, x) = 0, \quad j = 1, 2, \dots, r, \quad (4.8)$$

$$B_{n,r}((t-x)^{2r-j}, x) = O(n^{-r}\varphi^{2r-2j}(x)), \quad x \in [\frac{1}{n}, 1 - \frac{1}{n}], \quad j = 0, 1, 2, \dots, r. \quad (4.9)$$

Case 1. $x \in [\frac{1}{n}, 1 - \frac{1}{n}]$. By using Taylor expansion, we have

$$\begin{aligned}
&\bar{w}(x)(\bar{F}_n(x) - B_{n,r}(\bar{F}_n, x)) \\
&= \bar{w}(x) \sum_{j=1}^{2r-1} \frac{1}{(2r-j)!} B_{n,r}((t-x)^{2r-j}, x) \bar{F}_n^{(2r-j)}(x) \\
&\quad + \bar{w}(x) B_{n,r}(\frac{1}{(2r-j)!} \int_x^t (t-u)^{2r-1} \bar{F}_n^{(2r)}(u) du, x) \\
&:= I_1 + I_2.
\end{aligned}$$

By (3.2), (3.7) and (4.9), we have for $1 \leq j \leq r$, then

$$\frac{\bar{w}(x)\varphi^{2r-2j}(x)}{n^r} \bar{F}_n^{(2r-j)}(x) \leq \frac{C}{n^r}(\|\bar{w}\bar{F}_n\| + \|\bar{w}\varphi^{2r}\bar{F}_n^{(2r)}\|) \leq \frac{C}{n^r}(\|\bar{w}f\| + \|\bar{w}\varphi^{2r}f^{(2r)}\|), \quad (4.10)$$

By (4.8) and (4.10), we have

$$I_1 \leq \bar{w}(x) \sum_{j=1}^{r-1} \frac{1}{(2r-j)!} |B_{n,r}((t-x)^{2r-j}, x) \bar{F}_n^{(2r-j)}(x)| \leq \frac{C}{n^r}(\|\bar{w}f\| + \|\bar{w}\varphi^{2r}f^{(2r)}\|).$$

If u is between t and x , we have $\frac{|t-u|^{2r-1}}{\varphi^{2r}(u)} \leq \frac{|t-x|^{2r-1}}{\varphi^{2r}(x)}$. Then

$$\begin{aligned}
&|\bar{w}(x)B_{n,r}(\frac{1}{(2r-j)!} \int_x^t (t-u)^{2r-1} \bar{F}_n^{(2r)}(u) du, x)| \\
&\leq C\bar{w}(x) \sum_{i=0}^{r-1} \sum_{k=0}^{n_i} |C_i(n)| \int_x^{\frac{k}{n_i}} |(\frac{k}{n_i} - u)^{2r-1} \bar{F}_n^{(2r)}(u)| du p_{n_i,k}(x) \\
&= C\bar{w}(x) \sum_{i=0}^{r-1} \sum_{k=1}^{n_i-1} |C_i(n)| \int_x^{\frac{k}{n_i}} |(\frac{k}{n_i} - u)^{2r-1} \bar{F}_n^{(2r)}(u)| du p_{n_i,k}(x) \\
&\quad + C\bar{w}(x) \sum_{i=0}^{r-1} |C_i(n)|(1-x)^{n_i} \int_0^x u^{2r-1} |\bar{F}_n^{(2r)}(u)| du \\
&\quad + C\bar{w}(x) \sum_{i=0}^{r-1} |C_i(n)| x^{n_i} \int_x^1 (1-u)^{2r-1} |\bar{F}_n^{(2r)}(u)| du \\
&:= J_1 + J_2 + J_3.
\end{aligned}$$

We have

$$\begin{aligned}
J_1 &\leq C\bar{w}(x)\varphi^{-2r}(x) \sum_{i=0}^{r-1} \sum_{k/n_i \in A} |C_i(n)(\frac{k}{n_i} - x)^{2r-1}| \int_x^{\frac{k}{n_i}} \varphi^{2r}(v) |\bar{F}_n^{(2r)}(v)| dv p_{n_i,k}(x) \\
&\quad + C\bar{w}(x)\varphi^{-2r}(x) \sum_{i=0}^{r-1} \sum_{x'_2 \leq k/n_i \leq x'_3} |C_i(n)(\frac{k}{n_i} - x)^{2r-1}| \int_x^{\frac{k}{n_i}} \varphi^{2r}(v) |H^{(2r)}(v)| dv p_{n_i,k}(x) \\
&:= \sigma_1 + \sigma_2.
\end{aligned}$$

Analogously, we can get $\sigma_1 \leq \frac{C}{n^r}(\|\bar{w}f\| + \|\bar{w}\varphi^{2r}f^{(2r)}\|)$. We note that $|\varphi^{2r}(v)H^{(2r)}(v)| \leq \max(|\varphi^{2r}(x'_1)H^{(2r)}(x'_1)|, |\varphi^{2r}(x'_4)H^{(2r)}(x'_4)|) := |\varphi^{2r}(a)H^{(2r)}(a)|$, $H^{(2r)}(x)$ is a linear function.

If $x \in [x'_1, x'_4]$, then $\bar{w}(x) \leq \bar{w}(a)$. So, we have

$$\begin{aligned}
\sigma_2 &\leq C\bar{w}(a)\varphi^{2r}(a)|H^{(2r)}(a)|\varphi^{-2r}(x) \sum_{i=0}^{r-1} \sum_{k=1}^{n_i-1} |C_i(n)|(\frac{k}{n_i} - x)^{2r} p_{n_i,k}(x) \\
&\leq \frac{C}{n^r}(\|\bar{w}f\| + \|\bar{w}\varphi^{2r}f^{(2r)}\|),
\end{aligned}$$

If $x \notin [x'_1, x'_4]$, by $\bar{w}(a) > n_i^{-\frac{\alpha}{2}}$, we have

$$\begin{aligned}
\sigma_2 &\leq C\bar{w}(a)\varphi^{-2r}(a)|H^{(2r)}(a)| \sum_{i=0}^{r-1} \sum_{x'_2 \leq k/n_i \leq x'_3} n_i^{\frac{\alpha}{2}} |C_i(n)|(\frac{k}{n_i} - x)^{2r} p_{n_i,k}(x) \\
&\leq \frac{C}{n^r}(\|\bar{w}f\| + \|\bar{w}\varphi^{2r}f^{(2r)}\|).
\end{aligned}$$

For J_2 , we have

$$\begin{aligned}
J_2 &\leq C\|\bar{w}\varphi^{2r}\bar{F}_n^{(2r)}\|\bar{w}(x) \sum_{i=0}^{r-1} |C_i(n)|(1-x)^{n_i} \int_0^x u^{2r-1}\bar{w}^{-1}(u)\varphi^{-2r}(u)du \\
&\leq \frac{C}{n^r}(\|\bar{w}f\| + \|\bar{w}\varphi^{2r}f^{(2r)}\|).
\end{aligned}$$

Similarly, we have

$$J_3 \leq \frac{C}{n^r}(\|\bar{w}f\| + \|\bar{w}\varphi^{2r}f^{(2r)}\|).$$

By bringing these facts together, we have

$$\|\bar{w}(\bar{B}_{n,r}(f) - \bar{F}_n)\| \leq \frac{C}{n^r}(\|\bar{w}f\| + \|\bar{w}\varphi^{2r}f^{(2r)}\|).$$

Case 2. $x \in [0, \frac{1}{n}]$ (Similarly as $x \in [1 - \frac{1}{n}, 1]$). By using Taylor expansion, we have

$$\begin{aligned}
\bar{w}(x)|B_{n,r}(\bar{F}_n, x) - \bar{F}_n(x)| &\leq \frac{\bar{w}(x)}{r!} \sum_{i=0}^{r-1} |C_i(n)|B_{n_i}(\int_x^t |(t-u)^r \bar{F}_n^{(r+1)}(u)| du, x) \\
&\quad + \frac{\bar{w}(x)}{r!} \sum_{i=0}^{r-1} |C_i(n)|(1-x)^{n_i} \int_0^x u^{2r-1} |\bar{F}_n^{(r+1)}(u)| du \\
&:= J_1 + J_2.
\end{aligned}$$

$$\begin{aligned}
J_1 &\leq C\bar{w}(x) \sum_{i=0}^{r-1} \sum_{k=0}^{n_i} \int_x^{\frac{k}{n_i}} |C_i(n) (\frac{k}{n_i} - u)^r \bar{F}_n^{(r+1)}(u)| du p_{n_i,k}(x) \\
&:= C\bar{w}(x) \sum_{i=0}^{r-1} \sum_{k=1}^{n_i-1} \int_x^{\frac{k}{n_i}} |C_i(n) (\frac{k}{n_i} - u)^r \bar{F}_n^{(r+1)}(u)| du p_{n_i,k}(x) \\
&\quad + C\bar{w}(x) \sum_{i=0}^{r-1} |C_i(n)| x^{n_i} \int_x^1 (1-u)^r |\bar{F}_n^{(r+1)}(u)| du \\
&\quad + C\bar{w}(x) \sum_{i=0}^{r-1} |C_i(n)| (1-x)^{n_i} \int_0^x u^r |\bar{F}_n^{(r+1)}(u)| du \\
&:= I_1 + I_2 + I_3.
\end{aligned}$$

Analogously, we can get

$$\begin{aligned}
I_1 &\leq \frac{C}{n^r} (\|\bar{w}f\| + \|\bar{w}\varphi^{2r} f^{(2r)}\|), \\
I_2 &\leq \frac{C}{n^r} (\|\bar{w}f\| + \|\bar{w}\varphi^{2r} f^{(2r)}\|), \\
I_3 &\leq \frac{C}{n^r} (\|\bar{w}f\| + \|\bar{w}\varphi^{2r} f^{(2r)}\|). \\
J_1 &\leq \frac{C}{n^r} (\|\bar{w}f\| + \|\bar{w}\varphi^{2r} f^{(2r)}\|), \\
J_2 &\leq \frac{C}{n^r} (\|\bar{w}f\| + \|\bar{w}\varphi^{2r} f^{(2r)}\|).
\end{aligned} \tag{4.11}$$

So, we have

$$\|\bar{w}(\bar{B}_{n,r}(f) - \bar{F}_n)\| \leq \frac{C}{n^r} (\|\bar{w}f\| + \|\bar{w}\varphi^{2r} f^{(2r)}\|).$$

Then

$$\begin{aligned}
\|\bar{w}(\bar{B}_{n,r}(f) - f)\| &\leq \|\bar{w}(f - \bar{F}_n(f))\| + \|\bar{w}(\bar{F}_n(f) - \bar{B}_{n,r}(f))\| \\
&\leq \frac{C}{n^r} (\|\bar{w}f\| + \|\bar{w}\varphi^{2r} f^{(2r)}\|).
\end{aligned}$$

If $f \in C_{\bar{w}}$, there exists $g \in W_{\bar{w}}^{2r}$, by (2.4) and the first inequality of (2.5), we have

$$\begin{aligned}
\|\bar{w}(\bar{B}_{n,r}(f) - f)\| &\leq \|\bar{w}(f - g)\| + \|\bar{w}\bar{B}_{n,r}(f - g)\| + \|\bar{w}(g - \bar{B}_{n,r}(g))\| \\
&\leq C(\|\bar{w}(f - g)\| + \frac{1}{n^r} (\|\bar{w}g\| + \|\bar{w}\varphi^{2r} g^{(2r)}\|)) \\
&\leq C(\omega_{\varphi}^{2r}(f, n^{-1/2})_{\bar{w}} + n^{-r} \|\bar{w}f\|).
\end{aligned}$$

4.5. Proof of (2.6)

From the proof of (2.5), we actually have

$$\|\bar{w}(\bar{B}_{n,r}(f) - f)\| \leq CK_{2r,\varphi}(f, t^r)_{\bar{w}}.$$

Therefore, $K_{2r,\varphi}(f, n^{-r})_{\bar{w}} = O(t^\alpha)$ implies

$$\|\bar{w}(\bar{B}_{n,r}(f) - f)\| \leq (n^{-\alpha/2}).$$

By (2.3) and (2.4), we may choose g properly such that $\|\bar{w}\varphi^{2r}g^{(2r)}\| < \infty$ and

$$\begin{aligned}
\omega_{\varphi}^{2r}(f, n^{-1/2})_{\bar{w}} + \frac{\|\bar{w}f\|}{n^r} &\leq \|\bar{w}(\bar{B}_{n,r}(f) - f)\| + \frac{1}{n^r}(\|\bar{w}\varphi^{2r}\bar{B}_{n,r}^{(2r)}(f - g)\| \\
&\quad + \|\bar{w}\varphi^{2r}\bar{B}_{n,r}^{(2r)}(g)\|) + \frac{\|\bar{w}f\|}{n^r} \\
&\leq \|\bar{w}(f - \bar{B}_{n,r}(f))\| + \frac{\|\bar{w}f\|}{n^r} + C\left(\frac{k}{n}\right)^r(\|\bar{w}(f - g)\| \\
&\quad + k^{-r}\|\bar{w}\varphi^{2r}g^{(2r)}\| + k^{-r}\|\bar{w}f\|) \\
&\leq \|\bar{w}(f - \bar{B}_{n,r}(f))\| + C\left(\frac{k}{n}\right)^r(\omega_{\varphi}^{2r}(f, k^{-1/2})_{\bar{w}} \\
&\quad + k^{-r}\|\bar{w}f\|).
\end{aligned}$$

Hence, by [5], we obtain the converse inequality.

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